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Violent and slow relaxations to sinh-Poisson equilibrium state in two-dimensional point vortex system

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Abstract

Characteristics of numerically-obtained typical distributions of a two-dimensional point vortex system after a violent relaxation at positive and negative temperatures are examined. It is anticipated that like gravitational N-body systems, the point vortex system rapidly relaxes to a state near the thermal equilibrium state by the violent relaxation and after that the system evolves toward the genuine thermal equilibrium state driven by a collisional slow process (the slow relaxation). The time scale of the slow relaxation is proportional to the number of the point vortices. Namely, it takes a very long time to reach the final thermal equilibrium state. The detailed mechanism of the violent and slow relaxations are still unclear. In this paper, we examine the mechanism of the slow relaxation numerically. When the system temperature determined by the initial system energy is negative, the system evolves to a state consisting of many small areas with different temperatures by the violent relaxation. In this state, the vorticity is determined as a function of the stream function, which means that the motion of the vortices across an isosurface of the stream function is restricted. Due to this restriction, the collisional relaxation process following the violent relaxation is slow.

Keywords: Relaxation process; 2D point vortex system; Onsager theory; Negative temperature

1 Introduction

Characteristics of numerically-obtained typical distributions of a two-dimensional (2D) point vortex system after a violent relaxation at positive and negative temperatures are examined [7].

The point vortex model is a simple tool for investigations of 2D turbulence. In point vortex simulations, Biot-Savart integral is required to determine a velocity field from the distribution of the point vortices. Its calculation cost is proportional to N^2 where N is the total number of point vortices. Typically it may take several months for point vortex simulations using a normal PCs with 10^4 vortices. To overcome this difficulty, we use a GPU (Graphics Processing Unit) to accelerate the calculation of the Biot-Savart integral. GPU has multiple processing units that operate simultaneously and provides an on-site, facile supercomputing environment for a relatively low cost.

A motivation of this research is to understand the kinetic theory of the 2D point vortex system which describes the relaxation process toward a Boltzmann-type thermal equilibrium state, especially under the negative absolute temperature. It has been anticipated that like gravitational N-body systems, the point vortex system rapidly relaxes to a state near a thermal equilibrium state by the (collisionless) violent relaxation [2,5]. After the violent relaxation, the system evolves to the genuine thermal equilibrium state described by the sinh-Poisson equation [4] through a collisional slow relaxation. Its time scale can be proportional to the number of the point vortices. The point vortex gives a formal solution of the 2D inviscid Euler equation. However, it has been anticipated that there is an viscous effect due to the discrete, point-wise distribution of the vortices [1,3]. It is likely that the slow relaxation is driven by the viscous effect. In this paper, we numerically examine the mechanism of the slow relaxation following the violent relaxation.

When the system temperature determined by the initial system energy is negative, the system consists of many small areas with different temperatures after the

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violent relaxation. In this state, the vorticity is determined as a function of the stream function, in other words, $\nabla \cdot (u\omega) = 0$ where u is the velocity field and ω is the vorticity. This means that the motion of the vortices across an isosurface of the stream function is restricted. It is concluded that this may be a reason why the relaxation after the violent relaxation is slow.

2 Point vortex system

Let us consider a system consisting of $N/2$ positive and $N/2$ negative point vortices confined in a circular area with radius R

$$\omega(r,t) = \omega_+(r,t) + \omega_-(r,t), \tag{1}$$

$$\omega_+ \equiv \sum_{i=1}^{N/2} \Omega\delta(r - r_i), \tag{2}$$

$$\omega_- \equiv - \sum_{i=N/2+1}^{N} \Omega\delta(r - r_i). \tag{3}$$

The circulation of the i-th point vortex Ω_i locating at r_i is either Ω or $-\Omega$ where Ω is a positive constant. Equation (1) is a formal solution of the 2D inviscid, incompressible Euler equation:

$$\frac{\partial\omega}{\partial t} + \nabla \cdot (u\omega) = 0. \tag{4}$$

Motion of the point vortices at $r_i = (x_i, y_i)$ obeys the following equation of motion:

$$\Omega_i \frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \quad \Omega_i \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}, \tag{5}$$

or equivalently the Biot-Savart integral

$$\begin{aligned} \frac{dr_i}{dt} = &-\frac{1}{2\pi} \sum_{j \neq i} \Omega_j \frac{(r_i - r_j) \times \hat{z}}{|r_i - r_j|^2} \\ &- \frac{1}{2\pi} \sum_{j} \Omega_j \frac{(r_i - \bar{r}_j) \times \hat{z}}{|r_i - \bar{r}_j|^2}. \end{aligned} \tag{6}$$

Here, H is the Hamiltonian representing the total system energy

$$\begin{aligned} H = &-\frac{1}{4\pi} \sum_i \sum_{j \neq i} \Omega_i \Omega_j \ln |r_i - r_j| \\ &+ \frac{1}{4\pi} \sum_i \sum_j \Omega_i \Omega_j \ln |r_i - \bar{r}_j| \\ &- \frac{1}{4\pi} \sum_i \sum_j \Omega_i \Omega_j \ln \frac{R}{|r_j|} \end{aligned} \tag{7}$$

and $\hat{z}$ is the unit vector in z direction. The effect of the circular wall is introduced by the image vortex at $\bar{r}_i = R^2 r_i / |r_i|^2$.

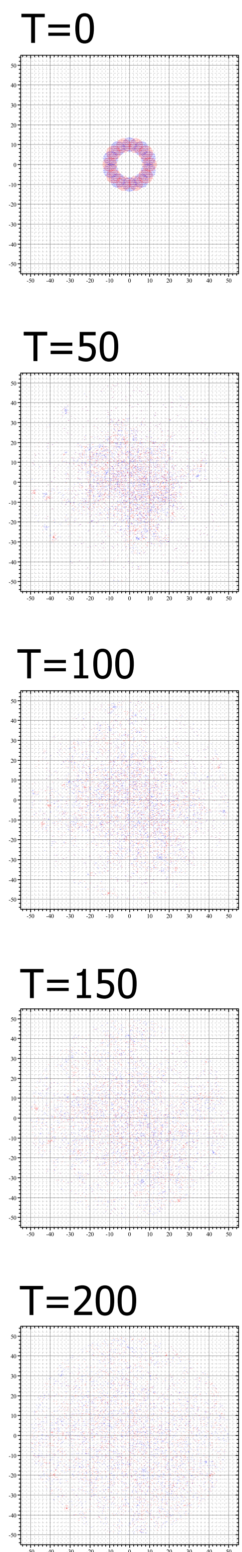

Figure 1 Typical time evolution of the point vortices at the positive temperature. Red and blue points correspond to the positive and negative point vortices, respectively. Initial distribution consists of 20 small clumps. As they overlap each other, the contour looks like a doughnut.

3 Negative absolute temperature of point vortex system

In 1949, Onsager proposed the negative temperature state for the point vortex system to understand the large-scale structure formation, in other words, the inverse-cascade [6]. By analogy between the usual canonical equation of motion and equation (5), phase space variables for the point vortex system are x_i and y_i as the configuration space and the phase space are identical. If N vortices are confined in a finite space with area A, the total phase space volume is limited to $A^N < \infty$, which implies that the density of state $W(E)$ equals zero as the system energy $E \to \infty$. Thus, the density of state has a peak at certain energy E_0 and dW/dE is negative at $E > E_0$, namely, the sign of the inverse temperature β is negative

$$\beta = \frac{d \log W(E)}{dE} < 0 \tag{8}$$

where the Boltzmann principle which defines the relation between the density of state and the entropy is assumed. This is a key mechanism that the 2D point vortex system confined in a finite area has a negative temperature state.

4 Simulation results

4.1 Typical distributions after violent relaxation

Distributions after the violent relaxation at positive and negative temperatures are obtained time asymptotically by numerical simulations using GPUs. Temperature is controlled by the initial system energy, namely, by the initial distribution of the vortices which is given artificially. Because of the violent relaxation, dependency on the initial artificial distribution is lost and after the violent relaxation, the distribution gradually approach the thermal equilibrium one. Typical time evolutions of the point vortices at the positive and negative temperatures are given in Figures 1 and 2.

At negative temperature like-sign vortices tend to cluster very rapidly by the violent relaxation and make two clumps exclusively consisting of the same-sign vortices. The self-rotation time scale of the clumps is approximately $T \sim 10$. At the beginning of the simulation, the clumps form within the time scale of $T \sim 10$ which is regarded as the time scale of the violent relaxation. After the violent relaxation, the clump distribution quasi-statically maintains up to $T = 200$ which is the end of the simulation. On the other hand, at positive temperature both types of vortices spread over the circular area uniformly. There is no rapid relaxation process as the value of the stream function is almost zero due to the uniform and symmetric distributions of the positive and negative vortices. Thus, the induced velocity is almost zero everywhere in the circular domain.

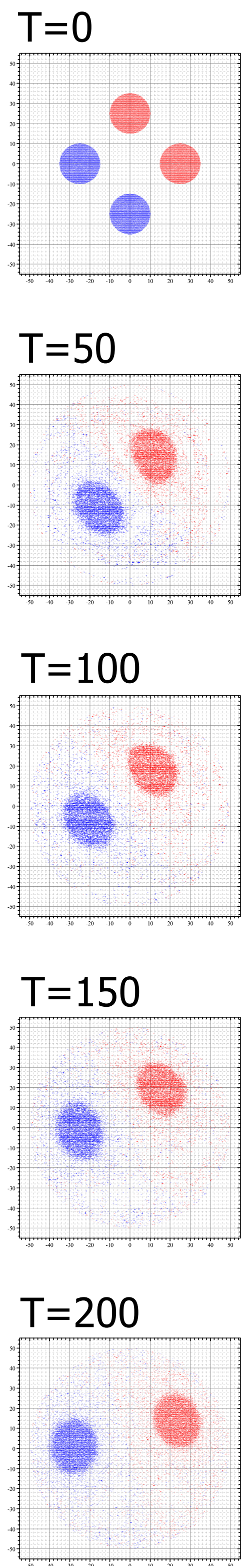

Figure 2 Typical time evolution of the point vortices at the negative temperature. As the system energy is high, the temperature can be negative and condensation of the same-sign vortices is possible.

4.2 Observation of local temperature via population of the vortices

Local temperature of the point vortex system is determined by a population (histogram) as a function of energy of each point vortex ψ_i defined by

$$H = \frac{1}{2}\sum_i \Omega_i \psi_i, \tag{9}$$

$$\begin{aligned}\psi_i = &-\frac{1}{2\pi}\sum_{j\neq i}\Omega_j \log|r_i - r_j| + \frac{1}{2\pi}\sum_j \Omega_j \log|r_i - \bar{r}_j| \\ &-\frac{1}{2\pi}\sum_j \Omega_j \log\frac{R}{|r_j|}\end{aligned} \tag{10}$$

Function ψ_i is the stream function at r_i which corresponds to the energy possessed by the i-th point vortex.

The reason why the temperature is observed by the population ψ_i is as follows. Let us introduce notations $N_+(E)$ and $N_-(E)$ that represent the number of the point vortices whose energy is in the range of $E \sim E + \Delta E$. The value of $N_\pm(E)$ with $E \approx \Omega\psi_i/2$ can be determined explicitly from a numerically-obtained distribution of the vortices.

On the other hand, $N_\pm(E)$ can be defined by

$$N_\pm(E) = \int_{D(E)} \frac{\omega_\pm(r,t)}{\pm\Omega} dr. \tag{11}$$

where a region in which the relation

$$E \le \frac{\pm\Omega\psi(r)}{2} \le E + \Delta E \tag{12}$$

is satisfied is denoted by $D(E)$. In double sign notation ($\pm$ and $\mp$) the upper sign corresponds to the other upper sign and vice versa. The value of the stream function satisfying the inequality (12) will be denoted by Ψ,

$$\Psi \approx \frac{2}{\pm\Omega}E. \tag{13}$$

In a quasi-stationary state, vorticity ω_+ and ω_- are functions of the stream function ψ, namely

$$\omega_\pm = \omega_\pm(\psi). \tag{14}$$

The evidence of the above relation will be demonstrated later. Inside the region $D(E)$, vorticity is given by $\omega_\pm = \omega_\pm(\Psi)$ that is approximately constant. Inserting this into equation (11), we obtain

$$\begin{aligned}N_\pm(E) &= \int_{D(E)} \frac{\omega_\pm(\Psi)}{\pm\Omega} dr \\ &= \frac{\omega_\pm(\Psi)}{\pm\Omega}\int_{D(E)} dr.\end{aligned} \tag{15}$$

The integral in the last formula gives the area of the region $D(E)$. The local equilibrium distribution is given by

$$\omega_\pm(\Psi) = \omega_{0\pm}\exp(\mp\beta(r)\Omega\Psi) \tag{16}$$

where $\omega_{0\pm}$ are constants and $\beta(r)$ is the (local) temperature at r. Inserting equation (16) into (15), we finally obtain

$$\frac{N_\pm(E)}{\int_{D(E)} dr} = \frac{\omega_{0\pm}}{\pm\Omega}\exp(\mp\beta(r)\Omega\Psi). \tag{17}$$

Equation (17) means that the population of the vortices as the function of energy of the point vortices is proportional to $\exp(\mp\beta(r)\Omega\Psi)$. This indicates that the local temperature at the position where the value of the stream function is evaluated can be determined by using the population as the function of each point vortex $\Psi = \psi_i$. We call $N_\pm(E)/\int_{D(E)} dr$ normalized population. As direct observation of the area $\int_{D(E)} dr$ is difficult, we replace it with the contour length of the vortex distribution. Normalized populations (histogram) as a function of ψ_i (a) at positive and (b) at negative temperatures are plotted in Figure 3.

At first we focus on the population of the positive vortices in negative temperature indicated by the black line in Figure 3(b). The population at $\psi_i < 0$ is very small and there may be no apparent physical meanings. As ψ_i increases, the population gradually decreases at $0 \le \psi_i \le 20$. This population indicates that the temperature of the background vortices outside the clump is locally positive, which will be confirmed in Figure 4. In the energy range $20 < \psi_i < 30$, the population increases linearly in log scale and β is almost constant. Finally, in the energy range $30 < \psi_i$, the slope is zero. As the vortices categorized in this energy region have large energy, it enables the formation of the clump with the same-sign vortices. This plot corresponds to the population heading towards the thermal equilibrium state. It may be expected that the population of the positive vortices has a positive slope of $-\beta\Omega > 0$ in all the energy ranges when the system reaches the thermal equilibrium state. Note that the value of β may be a unique constant (system parameter) at that time, although during the relaxation the system consists of the many small regions with different β.

The above discussion focuses on the positive vortices with the black line. The same discussion is valid for the negative vortices with the red line. Note that the Boltzmann factor changes into $\exp(\beta\Omega\psi_i)$ as the sign of Ω changes into $-\Omega$. Thus, the population of the negative

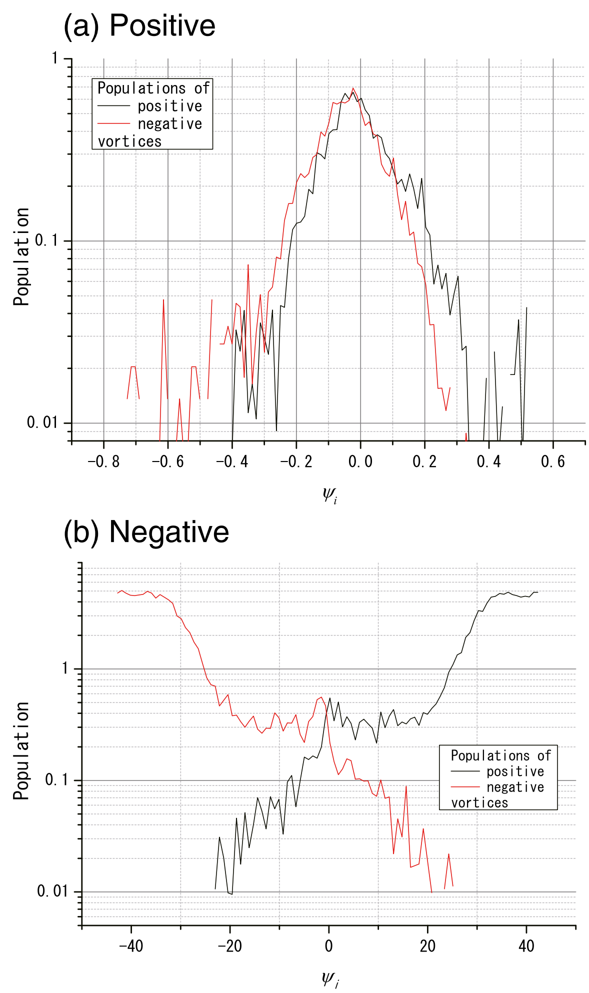

Figure 3 Populations for the equilibrium distributions (a) at positive and (b) at negative temperature are plotted as a function of energy possessed by each point vortex ψ_i.

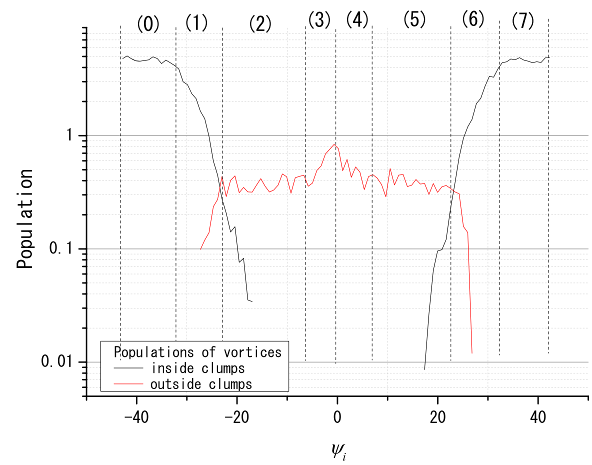

Figure 4 Populations for the negative temperature distribution are plotted separately inside and outside the clumps. Black and red lines correspond to the inside and outside the clumps, respectively.

vortices decreases as the energy increases in the case of negative temperature.

In Figure 3(a), the positive temperature case is shown. For the positive vortices, the slope is expected to be positive as the temperature can be positive. However, the population shows a symmetric profile around $\psi_i = 0$. The reason is unclear and further investigation is needed.

To confirm the origin of the peaks located at the both ends in Figure 3(b), the population is recalculated separately for the point vortices inside and outside the clumps. The result is shown in Figure 4.

The black line indicates the population of the vortices inside the clumps, and the red line outside the clumps. This shows the high energy vortices really locate inside the clumps.

Figure 5 shows the vortex distribution categorized by eight groups of the energy ranges in Figure 4.

In Figure 5, specifying a value of ψ_i equals selecting an area of certain color. As the isosurface of the stream function coincides with the boundary of the areas with different vorticity (different color). Namely, the figure implies that the vorticity ω is a function of the stream function ψ, $\omega = \omega(\psi)$ or equivalently $\nabla \cdot (u\omega) = 0$. If the evolution equation is given by the inviscid Euler equation, the equation is reduced to

$$\frac{\partial \omega}{\partial t} = 0 \tag{18}$$

and the distribution given in Figure 5 would be an equilibrium one. However, the kinetic theory for the 2D point vortex system anticipates the presence of the viscous effect of the order $1/N$ due to the discrete distribution of the vortices [1,3]. According to this prospect, there remains a small but finite collisional effect, even if the convective term $\nabla \cdot (u\omega)$ equals zero. This may be a driving source of the slow relaxation following the violent relaxation. The boundaries of the vortices with different temperatures are parallel to the isosurface of the stream function. As the flow (particle motion) across the isosurface of the stream function is very restricted, the subsystems with different temperature are preserved during the slow evolution and the system barely reaches the global thermal equilibrium state. This may be a reason for the slow relaxation.

We assume the typical temperature of this system is characterized by the regions (1) and (6) in Figure 4. The slopes in energy regions (1) and (6) are observed with various N keeping total circulation $N\Omega$ constant. The result is shown in Figure 6. The values of the slopes are

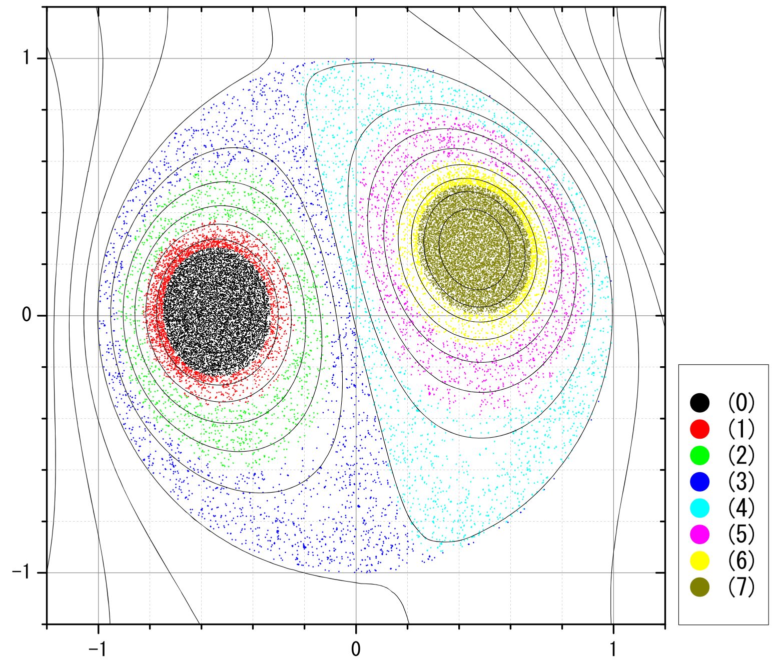

Figure 5 Vortex distribution is color-coded by the eight energy ranges indicated in Figure 4.

almost constant with different N. We conclude the slope corresponds to the system temperature.

5 Conclusion

Mechanism of the violent and slow relaxations is demonstrated by GPU numerical simulations. When temperature is negative, distribution evolves rapidly and two clumps consisting of positive and negative vortices exclusively is formed by the violent relaxation. After this relaxation, the system reaches the distribution with $\nabla \cdot (u\omega) = 0$ and then evolves slowly driven by the collisional effect which is anticipated by the kinetic theories for the 2D point vortex system. The effect is of the order of $1/N$ that results in the slow relaxation.

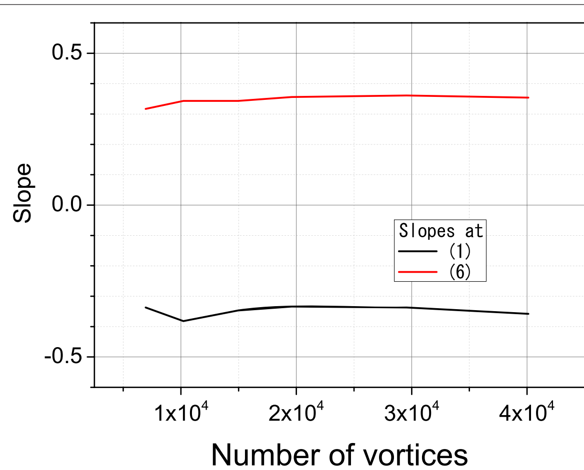

Figure 6 The slopes in energy regions (1) and (6) in Figure 3 are observed with various *N* keeping total circulation *N*Ω constant, which may correspond to the system temperature.

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