Fujita *et al. Pacific Journal of Mathematics for Industry* 2014, **6**:2
http://www.pacific-mathforindustry.com/content/6/1/2

Pacific Journal of Mathematics for Industry
a SpringerOpen Journal

ORIGINAL ARTICLE Open Access

On the one-sided maximum of Brownian and random walk fragments and its applications to new exotic options called "meander option"

Takahiko Fujita[1*], Yasuhiro Kawanishi[1] and Marc Yor[2^]

Abstract

We consider some distributions of one sided maxima of excursions and related variables for standard random walk and Brownian motion. We propose some new exotic options called meander options related to one of the fragments: the meander. We discuss the prices of meander options in a Black-Scholes market.

2000 AMS Subject classification: 60J55; 60J65; 91B28

Keywords: Standard random walk; Standard Brownian motion; Excursion; Meander; Exotic option; Black Scholes market

1 Introduction

1.1 Two-sided maxima for BW and RW

In our previous paper ([3]), distributions of

$$\sup_{t\leqq\theta}|B_t|,\quad \sup_{t\leqq g_\theta}|B_t|,\quad \sup_{t\leqq d_\theta}|B_t|$$

and

$$\sup_{t\leqq\theta}|Z_t|,\quad \sup_{t\leqq g_\theta}|Z_t|,\quad \sup_{t\leqq d_\theta}|Z_t|$$

were investigated where $(B_t, t \geqq 0)$ denotes Brownian Motion (written later as BM), whereas $(Z_t, t \in \mathbb{N} = \{0, 1, 2, \dots\})$ denotes a standard Random Walk (:RW), i.e: $Z_t = \xi_1 + \cdots + \xi_t$ where $\xi_1, \dots, \xi_t$ are *i.i.d.* and $P(\xi_1 = 1) = P(\xi_1 = -1) = 1/2$.

For $u \geqq 0$, $g_u = \sup\{s \leqq u : B_s = 0\}$, $d_u = \inf\{s > u : B_s = 0\}$ and $\theta \sim Exp(\frac{\lambda^2}{2})$ is independent of BM, whereas in the RW case for $u \in \mathbb{N}$, $g_u = \sup\{s \leqq u : Z_s = 0\}$, $d_u = \inf\{s > u : Z_s = 0\}$ and $\theta \sim \mathrm{Geom}(1-q)$ i.e: $P(\theta = k) = (1-q)q^k$ for $k \geqq 0$) is independent of RW.

In our previous paper, we also discussed some relations between the functional equation for the Riemann zeta function and the maximum of Brownian excursion, as well as some infinite divisibility properties of $d_\theta - g_\theta$, i.e.:

$$E\left(e^{-\mu(d_\theta-g_\theta)}\right) = \frac{\sqrt{m}}{\sqrt{\mu+m}+\sqrt{\mu}} = \exp\left(-\int_0^\infty (1-e^{-\mu x})\frac{dx}{2x}\int_0^x \frac{1}{\pi}\frac{e^{-my}}{\sqrt{y(x-y)}}dy\right)$$

where $\theta \sim \mathrm{Exp}(m)$, which means that the length $d_\theta - g_\theta$ of the excursion straddling θ is infinitely divisible and its Lévy Khintchin density is the Laplace transform of the arcsine law $\times \frac{1}{2x}$.

For more on these two topics, see, e.g. Biane-Pitman-Yor [2] and Bertoin-Fujita-Roynette-Yor [1].

In this paper, instead of two-sided maxima, we shall consider one sided maxima for these fragments and investigate their distributions.

1.2 New exotic options called "Meander Option"

Using these mathematical results, we consider some application for mathemtical finance. We define "meander options", the payoff of which is defined by the meander

*Correspondence: rankstatistics@gmail.com
^Deceased
[1] Faculty of Science and Engineering, Department of Industrial and Systems Engineering, Chuo University, 1-13-27, Kasuga, Bunkyo, Tokyo, Japan
Full list of author information is available at the end of the article

of stock value process above a strike price K before a maturity time T. For example,

- Payoff of "meander lookback call option" = $\max_{g_T^{(K)} \leqq u \leqq T} (S_u - K)^+$, where $x^+ = \max(x, 0)$ and S_t is a stock value process, $g_T^{(K)} = \sup\{t < T | S_t = K\}$.

The financial meaning of meander lookback option is the following: If we consider a usual lookback option (with payoff: $\max_{0 \leqq u \leqq T}(S_u - K)^+$ the price of this option is sometimes extremely high. So partial lookback option (with payoff: $\max_{u \in J}(S_u - K)^+$ where $J \subset [0, T]$ is considered and sometimes traded. Meander lookback option is one example of this partial lookback option with closed price formula.

1.3 Self-explanatory tables for computations

We now present our results in the form of two self-explanatory Tables.

1.4 Organisation of our paper

In section 2, we indicate how to obtain the formulae for the distributions of the six maxima in Table 1, either for BM or for RW.

In section 3, we do the same for Table 2. In section 4, we apply some of the above results to get the price of the meander lookback option; to do this, we first compute at independent exponential time, then invert the Laplace transform.

2 Computations of distributions for the six maxima

Notation: For clarity, we write: $P(\Gamma || \Lambda)$ for $\frac{P(\Gamma \cap \Lambda)}{P(\Lambda)}$ and $P(\Gamma || X = x)$ for the conditional law of Γ, given X.

$$\boxed{(\mathbf{1})\ \mathbf{P}\left(\sup_{u \leqq \theta} \mathbf{B}_u \leqq \mathbf{A}\right)}$$

$$\begin{aligned} P\left(\sup_{u \leqq \theta} B_u \leqq A\right) &= 1 - P\left(\sup_{u \leqq \theta} B_u \geqq A\right) = 1 - P\left(\theta \geqq T_A\right) \\ &= 1 - E\left(\exp\left(-\frac{\lambda^2 T_A}{2}\right)\right), \\ &\quad \text{where} \quad T_A = \inf\{t : B_t = A\} \\ &= 1 - e^{-\lambda A}. \end{aligned}$$

$$\boxed{(\mathbf{1'})\ \mathbf{P}\left(\sup_{u \leqq \theta} \mathbf{Z}_u < \mathbf{A}\right)}$$

$$\begin{aligned} P\left(\sup_{u \leqq \theta} Z_u < A\right) &= 1 - P\left(\sup_{u \leqq \theta} Z_u \geqq A\right) = 1 - P\left(\theta \geqq T_A\right) \\ &= 1 - E\left(q^{T^A}\right) \\ &\quad \text{where} \quad T_A = \inf\{t : Z_t = A\} \\ &= 1 - \alpha^A. \end{aligned}$$

Table 1 A list of interesting maxima

BM	$P(\cdot \leqq A)$	RW	$P(\cdot < A)$
$\sup_{u \leqq \theta} B_u \sim \sqrt{\theta} \sup_{u \leqq 1} B_u$	$1 - e^{-\lambda A}$	$\sup_{u \leqq \theta} Z_u$	$1 - \alpha^A$
$\sup_{u \leqq g_\theta} B_u \sim \sqrt{g_\theta} \sup_{u \leqq 1} b_u$ b : brownian bridge(b.b.)	$1 - e^{-2\lambda A}$	$\sup_{u \leqq g_\theta} Z_u$	$1 - \alpha^{2A}$
$\sup_{g_\theta \leqq u \leqq \theta} B_u \sim \epsilon\sqrt{\theta - g_\theta} \sup_{u \leqq 1} m_u$ m : brownian meander	$\frac{1}{1+e^{-\lambda A}}$	$\sup_{g_\theta \leqq u \leqq \theta} Z_u$	$\frac{1}{1+\alpha^A}$
$\sup_{u \leqq d_\theta} B_u$	$1 - \frac{1-e^{-2\lambda A}}{2\lambda A}$	$\sup_{u \leqq d_\theta} Z_u$	$1 - \frac{1}{A}\frac{1}{\alpha^{-1}-\alpha}\left(1 - \alpha^{2A}\right)$
$\sup_{\theta \leqq u \leqq d_\theta} B_u$	$1 - \frac{1-e^{-\lambda A}}{2\lambda A}$	$\sup_{\theta \leqq u \leqq d_\theta} Z_u$	$1 - \frac{2}{\alpha^{-1}-\alpha}\frac{1-\alpha^A}{A}$
$\sup_{g_\theta \leqq u \leqq d_\theta} B_u \sim \epsilon\sqrt{d_\theta - g_\theta} \sup_{u \leqq 1} e_u$ e : normalized excursion	$\frac{1}{1-e^{-2\lambda A}} - \frac{1}{2\lambda A}$	$\sup_{g_\theta \leqq u \leqq d_\theta} Z_u$	$\frac{1}{1-\alpha^{2A}} - \frac{1}{A}\frac{1}{\alpha^{-1}-\alpha}$

where
- for BM, $\theta \sim \text{Exp}(\lambda^2/2)$, *i.e.*, its density is $f_\theta(x) = 1_{(0,\infty)}(x)\frac{\lambda^2}{2}\exp - \frac{\lambda^2 x}{2}$, and $P(\epsilon = 1) = P(\epsilon = 0) = 1/2$.
- for RW, $\theta \sim \text{Geom}(1-q)$, *i.e.*, $P(\theta = k) = (1-q)q^k$, $(k = 0, 1, 2, \dots)$, $\alpha = \frac{1-\sqrt{1-q^2}}{q}$.
- for RW, $g_t = \sup\{u \leqq t : Z_u = 0\}$, $d_t = \inf\{u > t : Z_u = 0\}$.

Table 2 A list of joint distributions

BM	$P(\cdot \leqq A, B_\theta \in dx)$
$P\left(\sup_{u\leqq\theta} B_u \leqq A, B_\theta \in dx\right)$	$\left(\frac{\lambda}{2}e^{-\lambda\lvert x\rvert} - \frac{\lambda}{2}e^{\lambda x}e^{-2\lambda\max(A,x)}\right)dx$
$P\left(\sup_{u\leqq g_\theta} B_u \leqq A, B_\theta \in dx\right)^{*}$	$\frac{\lambda}{2}e^{-\lambda\lvert x\rvert}\left(1-e^{-2\lambda A}\right)dx$
$P\left(\sup_{g_\theta\leqq u\leqq\theta} B_u \leqq A, B_\theta \in dx\right)$	$\frac{\lambda}{2}\frac{1}{1-e^{-2\lambda A}}1_{x\leqq A}\left(e^{-\lambda\lvert x\rvert} - e^{\lambda x-2\lambda A}\right)dx$
$P\left(\sup_{u\leqq d_\theta} B_u \leqq A, B_\theta \in dx\right)$	$\left(1-\frac{x^+}{A}\right)1_{x\leqq A}\frac{\lambda}{2}\left(e^{-\lambda\lvert x\rvert} - e^{\lambda x-2\lambda A}\right)dx$
$P\left(\sup_{\theta\leqq u\leqq d_\theta} B_u \leqq A, B_\theta \in dx\right)$	$\left(1-\frac{x^+}{A}\right)1_{x\leqq A}\frac{\lambda}{2}e^{-\lambda\lvert x\rvert}dx$
$P\left(\sup_{g_\theta\leqq u\leqq d_\theta} B_u \leqq A, B_\theta \in dx\right)$	$\frac{1-\frac{x^+}{A}}{1-e^{-2\lambda A}}1_{x\leqq A}\frac{\lambda}{2}\left(e^{-\lambda\lvert x\rvert} - e^{\lambda x-2\lambda A}\right)dx$

(∗) Note: We see on this line that $\sup_{u\leqq g_\theta} B_u$ and B_θ are independent.

(2) $\mathbf{P}\left(\sup_{t\leqq g_\theta} \mathbf{B}_t \leqq \mathbf{A}\right)$

$$\begin{aligned}
P\left(\sup_{t\leqq g_\theta} B_t \leqq A\right) &= P\left(g_\theta \leqq T_A\right) = P\left(\theta \leqq d_{T_A}\right)\\
&= 1 - P(\theta \geqq d_{T_A})\\
&= 1 - E\left(\exp\left(-\frac{\lambda^2}{2}d_{T_A}\right)\right)\\
&= 1 - E\left(\exp\left(-\frac{\lambda^2}{2}T_A\right)\right)\\
&\quad \times E\left(\exp\left(-\frac{\lambda^2}{2}T_A\right)\right)\\
&\quad \text{where} \quad T_A = \inf\{t : B_t = A\}\\
&= 1 - e^{-2\lambda A}
\end{aligned}$$

(2′) $\mathbf{P}\left(\sup_{t\leqq g_\theta} \mathbf{Z}_t < \mathbf{A}\right)$

$$\begin{aligned}
P\left(\sup_{t\leqq g_\theta} Z_t < A\right) &= P\left(g_\theta < T_A\right) = P\left(\theta < d_{T_A}\right)\\
&= 1 - P\left(\theta \geqq d_{T_A}\right)\\
&= 1 - E\left(q^{d_{T_A}}\right) = 1 - E\left(q^{T_A}\right)E\left(q^{T_A}\right)\\
&= 1 - \alpha^{2A}.
\end{aligned}$$

(3) $\mathbf{P}\left(\sup_{g_\theta\leqq t\leqq\theta} \mathbf{B}_t \leqq \mathbf{A}\right)$

We start with:

$$P\left(\sup_{t\leqq g_\theta} B_t \leqq A\right)P\left(\sup_{g_\theta\leqq t\leqq\theta} B_t \leqq A\right) = P\left(\sup_{t\leqq\theta} B_t \leqq A\right),$$

since pre-g_θ events and post-g_θ events are independent. (see Revuz-[4], Chapter XII).

Then, we get:

$$P\left(\sup_{g_\theta\leqq t\leqq\theta} B_t \leqq A\right) = \frac{1-e^{-\lambda A}}{1-e^{-2\lambda A}} = \frac{1}{1+e^{-\lambda A}}$$

(3′) $\mathbf{P}\left(\sup_{g_\theta\leqq t\leqq\theta} \mathbf{Z}_t < \mathbf{A}\right)$

For random walk, we do similarly as with the preceding argument:

$$P\left(\sup_{t\leqq g_\theta} Z_t < A\right)P\left(\sup_{g_\theta\leqq t\leqq\theta} Z_t < A\right) = P\left(\sup_{t\leqq\theta} Z_t < A\right).$$

Then we get:

$$P\left(\sup_{g_\theta\leqq t\leqq\theta} Z_t < A\right) = \frac{1-\alpha^A}{1-\alpha^{2A}} = \frac{1}{1+\alpha^A}.$$

(4) $\mathbf{P}\left(\sup_{t\leqq d_\theta} \mathbf{B}_t \leqq \mathbf{A}\right)$

$$\begin{aligned}
P\left(\sup_{t\leqq d_\theta} B_t \leqq A\right) &= P\left(d_\theta \leqq T_A\right) = 1 - P\left(\theta > g_{T_A}\right)\\
&= 1 - E\left(\exp\left(-\lambda^2 g_{T_A}/2\right)\right)\\
&= 1 - \frac{e^{-\lambda A}}{\frac{\lambda A}{\sinh\lambda A}}\\
&= 1 - \frac{1-e^{-2\lambda A}}{2\lambda A},
\end{aligned}$$

since $E\left(\exp\left(\frac{-\lambda^2 g_{T_A}}{2}\right)\right)E\left(\exp\left(\frac{-\lambda^2}{2}(T_A - g_{T_A})\right)\right) = E\left(\exp\left(\frac{-\lambda^2}{2}T_A\right)\right)$ holds.

(4′) $\mathbf{P}\left(\sup_{t\leqq d_\theta} \mathbf{Z}_t < \mathbf{A}\right)$

$$\begin{aligned}
P\left(\sup_{t\leqq d_\theta} Z_t < A\right) &= P\left(d_\theta < T_A\right) = 1 - P\left(\theta \geqq g_{T_A}\right)\\
&= 1 - E\left(q^{g_{T_A}}\right)\\
&= 1 - \frac{\alpha^A}{A\frac{\alpha^{-1}-\alpha}{\alpha^{-A}-\alpha^A}}\\
&= 1 - \frac{1}{A}\frac{1}{\alpha^{-1}-\alpha}\left(1-\alpha^{2A}\right)
\end{aligned}$$

(5) $\mathrm{P}\left(\sup_{\theta \leqq t \leqq d_\theta} B_t \leqq A\right)$

If $x \leqq A$,

$$P\left(\sup_{\theta \leqq t \leqq d_\theta} B_t \leqq A || B_\theta = x\right) = 1 - \frac{x^+}{A},$$

where $x^+ := \max(x, 0)$.

Clearly if $A \leqq x$, $P\left(\sup_{\theta \leqq t \leqq d_\theta} B_t \leqq A || B_\theta = x\right) = 0$.

Then we get:

$$\begin{aligned} P\left(\sup_{\theta \leqq t \leqq d_\theta} B_t \leqq A\right) &= \int_0^A \left(1 - \frac{x^+}{A}\right) f_{B_\theta}(x)dx \\ &= \int_0^A \left(1 - \frac{x}{A}\right) \lambda e^{-\lambda x} dx \\ &= 1 - \frac{1 - e^{-\lambda A}}{\lambda A}. \end{aligned}$$

(5′) $\mathbf{P}\left(\sup_{\theta \leqq t \leqq d_\theta} \mathbf{Z}_t < \mathbf{A}\right)$

If $x \leqq A$,

$$\begin{aligned} P\left(\sup_{\theta \leqq t \leqq d_\theta} Z_t < A || Z_\theta = x\right) &= P\left(T_0 < T_{A-x}\right) \\ &= 1 - \frac{x}{A}. \end{aligned}$$

Clearly if $A \leqq x$, $P\left(\sup_{\theta \leqq t \leqq d_\theta} Z_t \leqq A || Z_\theta = x\right) = 0$.

Then we get:

$$\begin{aligned} P\left(\sup_{\theta \leqq t \leqq d_\theta} Z_t < A\right) &= \sum_{k=0}^{A} \left(1 - \frac{k}{A}\right) P\left(Z_\theta = k\right) \\ &= \frac{1}{1+\alpha} - \frac{1}{\alpha^{-1} - \alpha} \frac{1 - \alpha^A}{A} \end{aligned}$$

where we used the following facts:

$$\begin{aligned} E\left(t^{Z_\theta}\right) &= \sum_{k=0}^{\infty} \left(\frac{t + t^{-1}}{2}\right)^k (1-q) q^k \\ &= \frac{2(1-q)}{q\left(\alpha^{-1} - \alpha\right)} \left(\sum_{k=0}^{\infty} \left(\frac{\alpha}{t}\right)^{k+1} + \sum_{k=0}^{\infty} \alpha^k t^k\right). \end{aligned}$$

Then

$$P\left(Z_\theta = k\right) = \frac{1-\alpha}{1+\alpha} \alpha^k, \quad k \in \mathbb{Z}$$

and we see that

$$P\left(|Z_\theta| = k\right) = \begin{cases} \frac{1-\alpha}{1+\alpha} & \cdots\ k = 0 \\ \frac{2(1-\alpha)}{1+\alpha} \alpha^k & \cdots\ k \geqq 1 \end{cases}.$$

(6) $\mathbf{P}\left(\sup_{g_\theta \leqq t \leqq d_\theta} \mathbf{B}_t \leqq \mathbf{A}\right)$

$$\begin{aligned} P\left(\sup_{g_\theta \leqq t \leqq d_\theta} B_t \leqq A\right) &= \frac{P\left(\sup_{t \leqq d_\theta} B_t \leqq A\right)}{P\left(\sup_{t \leqq g_\theta} B_t \leqq A\right)} \\ &= \frac{1}{1 - e^{-2\lambda A}} - \frac{1}{2\lambda A}. \end{aligned}$$

(6′) $\mathbf{P}\left(\sup_{g_\theta \leqq t \leqq d_\theta} \mathbf{Z}_t < \mathbf{A}\right)$

$$\begin{aligned} P\left(\sup_{g_\theta \leqq t \leqq d_\theta} Z_t < A\right) &= \frac{P\left(\sup_{t \leqq d_\theta} Z_t < A\right)}{P\left(\sup_{t \leqq g_\theta} Z_t < A\right)} \\ &= \frac{1 - \frac{\alpha^A}{A \frac{\alpha^{-1} - \alpha}{\alpha^{-A} - \alpha^A}}}{1 - \alpha^{2A}} \\ &= \frac{1}{1 - \alpha^{2A}} - \frac{1}{A} \frac{1}{\alpha^{-1} - \alpha} \end{aligned}$$

3 Computations of joint distributions

(1) $\mathbf{P}\left(\sup_{u \leqq \theta} B_u \geqq \mathbf{A},\ \mathbf{B}_\theta \in \mathbf{dx}\right)$

$$\begin{aligned} P\left(\sup_{u \leqq \theta} B_u \geqq A,\ B_\theta \in dx\right) &= E(P(X \geqq A, X - Y \in dx | X)) \\ &= E\left(1_{X \geqq A} 1_{X \geqq x} \lambda e^{-\lambda(X-x)}\right) \\ &= \frac{\lambda}{2} e^{\lambda x} e^{-2\lambda \max(A,x)} dx \end{aligned}$$

where we put $X = \sup_{u \leqq \theta} B_u$, $Y = \sup_{u \leqq \theta} B_u - B_\theta$ and $X \sim Y \sim Exp(\lambda)$, X and Y are independent.

(2) $\mathbf{P}\left(\sup_{t \leqq g_\theta} \mathbf{B}_t \geqq \mathbf{A}, \mathbf{B}_\theta \in \mathbf{dx}\right)$

$$\begin{aligned} P\left(\sup_{t \leqq g_\theta} B_t \geqq A, B_\theta \in dx\right) &= P\left(\sup_{t \leqq g_\theta} B_t \geqq A\right) P\left(B_\theta \in dx\right) \\ &= e^{-2\lambda A} \frac{\lambda}{2} e^{-\lambda |x|} dx \end{aligned}$$

since pre-g_θ events and post-g_θ events are independent. (see Revuz -Yor[5], Chapter XII).

(3) $\mathbf{P}\left(\sup_{g_\theta \leqq t \leqq \theta} \mathbf{B}_t \leqq \mathbf{A}, \mathbf{B}_\theta \in \mathbf{dx}\right)$

We start with:

$$\begin{aligned} & P\left(\sup_{t \leqq g_\theta} B_t \leqq A\right) P\left(\sup_{g_\theta \leqq t \leqq \theta} B_t \leqq A, B_\theta \in dx\right) \\ &= P\left(\sup_{t \leqq \theta} B_t \leqq A, B_\theta \in dx\right), \end{aligned}$$

since pre-g_θ events and post-g_θ events are independent.

Then, we get:

$$P\left(\sup_{g_\theta \leqq t \leqq \theta} B_t \leqq A, B_\theta \in dx\right) = \frac{\lambda}{2}\frac{1}{1-e^{-2\lambda A}} 1_{x \leqq A}\left(e^{-\lambda|x|} - e^{\lambda x - 2\lambda A}\right) dx$$

(4) $\mathbf{P}\left(\sup_{t \leqq d_\theta} \mathbf{B}_t \leqq \mathbf{A}, \mathbf{B}_\theta \in \mathbf{dx}\right)$

$$\begin{aligned}
P\left(\sup_{t \leqq d_\theta} B_t \leqq A, B_\theta \in dx\right) &= P\left(\sup_{t \leqq \theta} B_t \leqq A, \sup_{\theta \leqq t \leqq d_\theta} B_t \leqq A, B_\theta \in dx\right) \\
&= P\left(\sup_{\theta \leqq t \leqq d_\theta} B_t \leqq A || B_\theta = x, \sup_{t \leqq \theta} B_t \leqq A\right) \\
&\quad \times P\left(B_\theta \in dx, \sup_{t \leqq \theta} B_t \leqq A\right) \\
&= P\left(\sup_{\theta \leqq t \leqq d_\theta} B_t \leqq A || B_\theta = x\right) P\left(B_\theta \in dx, \sup_{t \leqq \theta} B_t \leqq A\right) \\
&= \left(1 - \frac{x^+}{A}\right) 1_{x \leqq A} \frac{\lambda}{2}\left(e^{-\lambda|x|} - e^{\lambda x - 2\lambda A}\right) dx
\end{aligned}$$

by the Markov property at θ.

(5) $\mathbf{P}\left(\sup_{\theta \leqq t \leqq d_\theta} \mathbf{B}_t \leqq \mathbf{A}, \mathbf{B}_\theta \in \mathbf{dx}\right)$

If $x \leqq A$,

$$\begin{aligned}
P\left(\sup_{\theta \leqq t \leqq d_\theta} B_t \leqq A, B_\theta \in dx\right) &= P\left(\sup_{\theta \leqq t \leqq d_\theta} B_t \leqq A || B_\theta = x\right) P\left(B_\theta \in dx\right) \\
&= \left(1 - \frac{x^+}{A}\right) \frac{\lambda}{2} e^{-\lambda|x|} dx.
\end{aligned}$$

If $x > A$, the result is trivially 0.

(6) $\mathbf{P}\left(\sup_{g_\theta \leqq t \leqq d_\theta} \mathbf{B}_t \leqq \mathbf{A}, \mathbf{B}_\theta \in \mathbf{dx}\right)$

$$\begin{aligned}
P\left(\sup_{g_\theta \leqq t \leqq d_\theta} B_t \leqq A, B_\theta \in dx\right) &= \frac{P\left(\sup_{t \leqq d_\theta} B_t \leqq A, B_\theta \in dx\right)}{P\left(\sup_{t \leqq g_\theta} B_t \leqq A\right)} \\
&= \frac{\left(1 - \frac{x^+}{A}\right)}{1 - e^{-2\lambda A}} 1_{x \leqq A} \frac{\lambda}{2}\left(e^{-\lambda|x|} - e^{\lambda x - 2\lambda A}\right) dx.
\end{aligned}$$

In the following section, we state applications of these exact computations to price some exotic options which we call "Meander Options".

4 Price of some meander options

4.1 Option price at independent exponential time

We consider the following Black Scholes Model under the risk neutral measure Q:

$$dS_t = rS_t dt + \sigma S_t dW_t, \quad S_0 = S$$

where S_t is the stock value at time t, r is the risk free rate, and σ is the volatility.

We get:

$$S_t = S \exp\left(\left(r - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right)$$

Then the risk neutral valuation for derivative with pay-off Y at maturity time T gives $V_0(Y)$, the present value of derivative Y:

$$V_0 = E\left(e^{-rT} Y\right)$$

If Y is of the form $\phi(F_T)$, instead of fixed time T, it may be more convenient to work at time θ, an independent exponential time, because using such θ often makes expressions simpler than at fixed time T.

There are 2 ways to access such results.

First attitude:

a) to obtain the law of F_t;
   in fact, very often for this, it is simpler to consider F_θ, $\theta \sim \mathrm{Exp}(\lambda)$, and to invert the Laplace transform to get the law of F_t. Then, compute $E(\phi(F_t))$ for the particular ϕ of interest.
b) second attitude: Start directly with

$$\lambda \int_0^\infty e^{-\lambda t} E\left(\phi\left(F_t\right)\right) dt = E\left(\phi\left(F_\theta\right)\right)$$

and invert the Laplace transform.

In fact, there is the commutative diagram:

$$\begin{array}{ccc}
\text{Law of } (F_\theta) & \longrightarrow & E\left(\phi\left(F_\theta\right)\right) \\
\downarrow & & \downarrow \\
\text{Law of } (F_t) & \longrightarrow & E\left(\phi\left(F_t\right)\right)
\end{array}$$

which indicates that we may use either route from NW to SE.

First we consider the case $\phi = f(S_T)$ which is only dependent on the final stock value S_T.

$$\begin{aligned}
C &= E\left(e^{-r\theta} f\left(Se^{(r-\frac{1}{2}\sigma^2)\theta+\sigma W_\theta}\right)\right)\\
&= E\left(\exp\left(\frac{r-\frac{1}{2}\sigma^2}{\sigma}W_\theta - \left(\frac{1}{2}\left(\frac{r-\frac{1}{2}\sigma^2}{\sigma}\right)^2 + r\right)\theta\right) f\left(Se^{\sigma W_\theta}\right)\right)\\
&\qquad (\because \text{Cameron-Martin})\\
&= E\left(\exp\left(-\frac{1}{2}\left(\frac{r}{\sigma}+\frac{\sigma}{2}\right)^2\theta\right)\exp\left(\frac{r-\frac{1}{2}\sigma^2}{\sigma}W_\theta\right) f\left(Se^{\sigma W_\theta}\right)\right)\\
&= \frac{\lambda}{\lambda'} E\left(\exp\left(\frac{r-\frac{1}{2}\sigma^2}{\sigma}W_{\theta'}\right) f\left(Se^{\sigma W_{\theta'}}\right)\right)\\
&\left(\text{where } \theta' \sim Exp\left(\lambda+\frac{1}{2}\left(\frac{r}{\sigma}+\frac{\sigma}{2}\right)^2\right) \quad \lambda' = \lambda+\frac{1}{2}\left(\frac{r}{\sigma}+\frac{\sigma}{2}\right)^2\right)\\
&= \frac{\lambda}{\lambda'}\int_{-\infty}^{\infty} e^{\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)x} f\left(Se^{\sigma x}\right)\frac{\sqrt{2\lambda'}}{2}e^{-\sqrt{2\lambda'}|x|}dx
\end{aligned}$$

Generally we get that $E(e^{-\alpha\theta} f(W_\theta)) = \int_0^\infty e^{-\alpha t} E(f(W_t))\lambda e^{-\lambda t}dt = \frac{\lambda}{\lambda+\alpha}E(f(W_{\theta'}))$ where we used that for $\theta \sim \mathrm{Exp}(\lambda)$, then $\theta' \sim \mathrm{Exp}(\lambda+\alpha)$.

We also used the simple facts $E\left(e^{\alpha W_{\theta'}}\right) = E\left(E\left(e^{\alpha W_{\theta'}}||\theta'\right)\right) = E\left(e^{\frac{\alpha^2\theta'}{2}}\right) = \frac{2\lambda'}{2\lambda'-\alpha^2} = \int_{-\infty}^{\infty} e^{\alpha x}\frac{\sqrt{2\lambda'}}{2}e^{-\sqrt{2\lambda'}|x|}dx$ then, we get

$$f_{W_{\theta'}}(x) = \frac{\sqrt{2\lambda'}}{2}e^{-\sqrt{2\lambda'}|x|}.$$

In the case of a call option, $f(S_T) = (S_T - K)^+$. We want to get the call option price when $K \geqq S$.

$$\begin{aligned}
C &= \frac{\lambda}{\lambda'}\int_{\frac{\log K/S}{\sigma}}^{\infty} e^{\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)x}\left(Se^{\sigma x}-K\right)\frac{\sqrt{2\lambda'}}{2}e^{-\sqrt{2\lambda'}|x|}dx\\
&= \frac{\lambda}{\lambda'}\int_{\frac{\log K/S}{\sigma}}^{\infty} S\frac{\sqrt{2\lambda'}}{2}e^{-\left(\sqrt{2\lambda'}-\sigma-\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)\right)x}dx\\
&\quad - \frac{\lambda}{\lambda'}\int_{\frac{\log K/S}{\sigma}}^{\infty} K\frac{\sqrt{2\lambda'}}{2}e^{-\left(\sqrt{2\lambda'}-\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)\right)x}dx\\
&= \frac{\lambda\sigma}{\sqrt{2\lambda'}\left(\left(\sqrt{2\lambda'}-\frac{r}{\sigma}\right)^2-\frac{\sigma^2}{4}\right)} S^{\frac{2\lambda'}{\sigma}-\frac{r}{\sigma^2}+\frac{1}{2}} K^{-\frac{2\lambda'}{\sigma}+\frac{r}{\sigma^2}+\frac{1}{2}}
\end{aligned}$$

We get the usual Black-Scholes formula by inverting the above with respect to λ.

4.2 Price of meander lookback option

$$\begin{aligned}
&V_0(\text{Meander lookback option up to time } \theta)\\
&\quad = E\left(e^{-r\theta}\max_{g_\theta^{(K)}\leqq u\leqq\theta} Se^{\exp((r-\frac{1}{2}\sigma^2)u+\sigma W_u)} - K\right)^+\Bigg).
\end{aligned}$$

In the following, we calculate the above in two cases: a) $\underline{S \leqq K}$ and b) $\underline{S \geqq K}$

a) $\underline{S \leqq K}$

$$\begin{aligned}
&E\left(e^{-r\theta} f\left(\max_{g_\theta^K\leqq t\leqq\theta} S_t\right)\right)\\
&= E\left(e^{-r\theta} f\left(\max_{g_\theta^K\leqq t\leqq\theta} S_t\right), \theta \geqq \tau_K\right)\\
&\quad + E\left(e^{-r\theta} f\left(\max_{g_\theta^K\leqq t\leqq\theta} S_t\right), \theta < \tau_K\right)\\
&= E\left(e^{-r\tau_K}\right) E\left(e^{-r\theta} f\left(\max_{g_\theta^K\leqq t\leqq\theta} Ke^{\sigma W_t+(r-\frac{1}{2}\sigma^2)t}\right)\right)\\
&\qquad \text{(by memoryless property)}\\
&= E\left(e^{-r\tau_K}\right) E\left(e^{-r\theta} e^{\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)W_\theta-\frac{1}{2}\left(\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)^2\theta\right.}\right.\\
&\qquad \left.\times f\left(\max_{g_\theta^K\leqq t\leqq\theta} Ke^{\sigma W_t}\right)\right) \quad \text{(by Cameron-Martin)}\\
&= E\left(e^{-r\tau_K}\right)\frac{\lambda}{\lambda'}E\left(e^{\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)W_{\theta'}} f\left(\max_{g_{\theta'}^K\leqq t\leqq\theta'} Ke^{\sigma W_t}\right)\right)\\
&= \frac{\lambda}{\lambda'}E\left(e^{-r\tau_K}\right)\int\int_{A\geqq x\geqq 0} e^{\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)x} f\left(Ke^{\sigma A}\right)\\
&\qquad \times \frac{\partial}{\partial A}\left(\frac{\lambda}{2}\frac{1}{1-e^{-2\lambda A}}\left(e^{-\lambda x}-e^{\lambda x-2\lambda A}\right)\right)dxdA\\
&= \frac{\lambda}{\lambda'}E\left(e^{-r\tau_K}\right)\int\int_{A\geqq x\geqq 0} e^{\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)x} f\left(Ke^{\sigma A}\right)\\
&\qquad \times \frac{\lambda^2}{4}\frac{\sinh\lambda x}{(\sinh\lambda A)^2}dxdA \quad (*)
\end{aligned}$$

b) $\underline{S \geqq K}$

$$\begin{aligned}
&E\left(e^{-r\theta} f\left(\max_{g_\theta^K\leqq t\leqq\theta} S_t\right)\right)\\
&= E\left(e^{-r\theta} f\left(\max_{g_\theta^K\leqq t\leqq\theta} S_t\right), \theta \geqq \tau_K\right) + E\left(e^{-r\theta} f\left(\max_{g_\theta^K\leqq t\leqq\theta} S_t\right), \theta < \tau_K\right)\\
&= E\left(e^{-r\tau_K}\right) E\left(e^{-r\theta} f\left(\max_{g_\theta^K\leqq t\leqq\theta} Ke^{\sigma W_t+(r-\frac{1}{2}\sigma^2)t}\right)\right)\\
&\quad + E\left(e^{-r\theta} f\left(\max_{g_\theta^K\leqq t\leqq\theta} S_t\right)\right), \min_{g_\theta^K\leqq t\leqq\theta} S_t \geqq K\right)\\
&= (*) + \frac{\lambda}{\lambda'}\int_0^\infty dbf\left(Se^{\sigma b}\right)\\
&\quad \times\left(-\frac{\partial}{\partial b}\frac{e^{\left(\frac{r}{\sigma^2}-\frac{1}{2}\right)\log\frac{K}{S}}\sinh\left(b\sqrt{2\lambda+\mu^2}\right)+e^{\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)b}\sinh-\frac{1}{\sigma}\log\frac{K}{S}\sqrt{2\lambda+\mu^2}}{\sinh\left(\left(b-\frac{1}{\sigma}\log\frac{K}{S}\right)\sqrt{2\lambda+\mu^2}\right.}\right)
\end{aligned}$$

For call option i.e. $f(x) = (x-K)^+$, we obtain that by some elementary calculation,

a) when $\underline{S \leqq K}$, the price equals:

$$\frac{K}{8\left(\lambda+\frac{1}{2}\left(\frac{r}{\sigma}+\frac{\sigma}{2}\right)^2\right)}\left(\frac{K}{S}\right)^{\frac{r}{\sigma^2}-\frac{1}{2}-\sqrt{\left(\frac{1}{2}-\frac{r}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}}}$$

$$\times\,\zeta^{(3)}\left(\frac{2\lambda-\sigma}{2\lambda},\frac{\lambda-\frac{\sigma}{2}-\frac{r}{\sigma}}{2\lambda},\frac{3\lambda-\frac{\sigma}{2}-\frac{r}{\sigma}}{2\lambda}\right) \qquad (**)$$

where $\zeta^{(3)}(A,B,C) := \sum_{l=0}^{\infty}\frac{1}{(l+A)(l+B)(l+C)}$.

Especially, if $\sigma^2 = 2r$, the price equals $\frac{K}{8\left(\lambda+\frac{\sigma^2}{2}\right)}\left(\frac{K}{S}\right)^{\frac{\sqrt{2\lambda}}{\sigma^2}}\zeta^{(3)}\left(\frac{\lambda-\sigma}{2\lambda},\frac{2\lambda-\sigma}{2\lambda},\frac{3\lambda-\sigma}{2\lambda}\right)$.

b) when $\underline{S \geqq K}$, the price equals: $(**)+\frac{\lambda}{\lambda+\frac{1}{2}\left(\frac{r}{\sigma}+\frac{\sigma}{2}\right)^2}$

$$\left(\left(1-\left(\frac{K}{S}\right)^{\left(\frac{r}{\sigma^2}-\frac{1}{2}\right)}\right)(S-K)+\sigma S\int_0^{\infty}e^{\sigma b}\left(\frac{e^{\left(\frac{r}{\sigma^2}-\frac{1}{2}\right)\log\frac{K}{S}}\sinh b\sqrt{2\lambda+\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)^2}+e^{\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)b}\sinh\left(-\frac{\sqrt{2\lambda+\left(\frac{r}{\sigma}-\frac{\sigma^2}{2}\right)^2}}{\sigma}\log\frac{K}{S}\right)}{\sinh\left(\left(b-\frac{1}{\sigma}\log\frac{K}{S}\right)\sqrt{2\lambda+\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)^2}\right)}-1\right)db\right).$$

Condolences

Prof. Marc Yor passed away suddenly on January 9 2014. He brought so many gifts to our mathematics. We will never forget him.

Author details

[1]Faculty of Science and Engineering, Department of Industrial and Systems Engineering, Chuo University, 1-13-27, Kasuga, Bunkyo, Tokyo, Japan.
[2]Laboratoire de Probabilités et Modèles Aléatoires, Université Paris VI, casier 188, 4, place Jussieu, 75252 Paris cedex 05, France.

Received: 28 March 2014 Accepted: 9 April 2014
Published online: 20 September 2014

References

1. Bertoin, J., Fujita, T., Roynette, B., Yor, M.: On a particular class of self-decomposable random variables: the durations of a Bessel excursion straddling an independent exponential time. Prob. Math. Stat. **26**(2), 315–366 (2006)
2. Biane, P., Pitman, J., Yor, M.: Probability laws related to Riemann zeta and theta functions. Bull. AMS. **38**, 435–469 (2001)
3. Fujita, T., Yor, M.: On the remarkable distributions of maxima of some fragments of the standard reflecting random walk and Brownian Motion. Prob. Math. Stat. **27**(1), 89–104 (2007)
4. Revuz, D., Yor, M.: Continuous Martingales and Brownian Motion, Third edition. Springer Velrag (2005)
5. Yor, M., Chesney, M., Geman, H., Jeanblanc-Picqué, M.: Some combinations of Asian, Parisian and barrier options. In: Dempster, MH., Pliska, SR. (eds.) Mathematics of derivative securities, pp. 61–87. Cambridge University Press, UK, (1997)

doi:10.1186/s40736-014-0002-0
Cite this article as: Fujita *et al.*: On the one-sided maximum of Brownian and random walk fragments and its applications to new exotic options called "meander option". *Pacific Journal of Mathematics for Industry* 2014 **6**:2.